

**JACOB'S LADDERS, GRAM'S SEQUENCE AND SOME  
NONLINEAR INTEGRAL EQUATIONS CONNECTED WITH  
THE FUNCTIONS  $J_\nu(x)$  AND  $|\zeta(\frac{1}{2} + it)|^4$**

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ABSTRACT. It is shown in this paper that the Jacob's ladder is the asymptotic solution to the new nonlinear integral equations which correspond to the functions  $J_\nu(x)$  and  $|\zeta(\frac{1}{2} + it)|^4$ .

1. THE FIRST RESULT: THE NONLINEAR INTEGRAL EQUATION CONNECTED WITH  
THE BESSEL'S FUNCTIONS

1.1. In this paper we obtain some new properties of the signal

$$Z(t) = e^{i\vartheta(t)} \zeta\left(\frac{1}{2} + it\right)$$

that is generated by the Riemann zeta-function, where

$$\vartheta(t) = -\frac{t}{2} \ln \pi + \operatorname{Im} \ln \Gamma\left(\frac{1}{4} + i\frac{t}{2}\right) = \frac{t}{2} \ln \frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8} + \mathcal{O}\left(\frac{1}{t}\right).$$

Let us remind that

$$\tilde{Z}^2(t) = \frac{d\varphi_1(t)}{dt}, \quad \varphi_1(t) = \frac{1}{2}\varphi(t)$$

where

$$(1.1) \quad \tilde{Z}^2(t) = \frac{Z^2(t)}{2\Phi'_\varphi[\varphi(t)]} = \frac{|\zeta(\frac{1}{2} + it)|^2}{\{1 + \mathcal{O}(\frac{\ln \ln t}{\ln t})\} \ln t}$$

(see [1], (3.9); [2], (1.3); [7], (1.1), (3.1), (3.2)), and  $\varphi(t)$  is the Jacob's ladder, i.e. a solution of the nonlinear integral equation (see [1])

$$\int_0^{\mu[x(T)]} Z^2(t) e^{-\frac{2}{x(T)}t} dt = \int_0^T Z^2(t) dt.$$

1.2. The Gram's sequence  $\{t_\nu\}$  is defined by the equation

$$\vartheta(t_\nu) = \pi\nu, \quad \nu = 1, 2, \dots$$

where (see [20], p. 102)

$$(1.2) \quad t_{\nu+1} - t_\nu = \frac{2\pi}{\ln t_\nu} + \frac{2\pi \ln 2\pi}{\ln^2 t_\nu} + \mathcal{O}\left(\frac{1}{\ln^3 t_\nu}\right).$$

The following theorem holds true.

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*Key words and phrases.* Riemann zeta-function.

**Theorem 1.** Every Jacob's ladder  $\varphi_1(t) = \frac{1}{2}\varphi(t)$  where  $\varphi(t)$  is the exact solution to the nonlinear integral equation

$$\int_0^{\mu[x(T)]} Z^2(t) e^{-\frac{2}{x(T)}t} dt = \int_0^T Z^2(t) dt$$

is the asymptotic solution of the following nonlinear integral equation

$$(1.3) \quad \int_{x^{-1}(t_\nu)}^{x^{-1}(t_{\nu+1})} J_1[x(t)] \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 dt = \frac{2\sqrt{2\pi}}{\sqrt{t_\nu}} \sin\left(t_\nu - \frac{\pi}{4}\right)$$

for every sufficiently big  $t_\nu$  that fulfils the conditions

$$(1.4) \quad \begin{aligned} [t_\nu, t_{\nu+1}] &\subset [\mu_n^{(1)}, \mu_{n+1}^{(1)}], \\ [t_\nu, t_{\nu+1}] \cap [k\pi - \epsilon, k\pi + \epsilon] &= \emptyset, \quad \nu, k \in \mathbb{N}, \quad \nu \rightarrow \infty \end{aligned}$$

where  $J_1(\mu_n^{(1)}) = 0$ ,  $n = 1, 2, \dots$ , i.e. the following asymptotic formula

$$(1.5) \quad \int_{\varphi_1^{-1}(t_\nu)}^{\varphi_1^{-1}(t_{\nu+1})} J_1[\varphi_1(t)] \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 dt \sim \frac{2\sqrt{2\pi}}{\sqrt{t_\nu}} \sin\left(t_\nu - \frac{\pi}{4}\right)$$

holds true.

*Remark 1.* Since

$$(1.6) \quad \mu_{n+1}^{(1)} - \mu_n^{(1)} \sim \pi, \quad n \rightarrow \infty$$

then the number  $N_{\nu,n}$  of the intervals  $[t_\nu, t_{\nu+1}]$ , for which the first condition in (1.4) is fulfilled, is given by the asymptotic formula

$$N_{\nu,n} \sim \frac{1}{2} \ln t_\nu, \quad t_\nu \rightarrow \infty,$$

((1.2), (1.6)).

This paper is a continuation of the series [1] - [19].

## 2. THE SECOND RESULT: SOME NONLINEAR INTEGRAL EQUATION CONNECTED WITH THE FUNCTION $|\zeta(\frac{1}{2} + it)|^4$

Let us remind that the Jacob's ladder  $\varphi_2(T)$  of the second order is a solution to the nonlinear integral equation

$$(2.1) \quad \int_0^{\mu[x(T)]} Z^4(t) e^{-\frac{t}{x(T)}} dt = \int_0^T Z^4(t) dt$$

(see [8]). In this case the following asymptotic formula (see [8], (1.5))

$$(2.2) \quad \begin{aligned} &\int_{\varphi_1^{-1}(T)}^{\varphi_1^{-1}(T+U)} \left| \zeta\left(\frac{1}{2} + i\varphi_2(t)\right) \right|^4 \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 dt \sim \\ &\sim \frac{1}{4\pi^4} U \ln^8 T, \quad U = T^{13/14+2\epsilon}, \quad T \rightarrow \infty \end{aligned}$$

holds true.

*Remark 2.* The small improvements of the exponent  $\frac{13}{14}$  that are of the type  $\frac{13}{14} \rightarrow \frac{8}{9} \rightarrow \dots$  are irrelevant in this question.

Next, similarly to the Theorem 1, the following theorem holds true.

**Theorem 2.** *Every Jacob's ladder of the second order  $\varphi_2(t)$ , i.e. the (exact) solution to the nonlinear integral equation (2.1) is the asymptotic solution of the nonlinear integral equation*

$$(2.3) \quad \int_{x^{-1}(T)}^{x^{-1}(T+U)} \left| \zeta \left( \frac{1}{2} + ix(t) \right) \right|^4 \left| \zeta \left( \frac{1}{2} + it \right) \right|^4 dt = \frac{1}{4\pi^4} U \ln^8 T, \quad T \rightarrow \infty,$$

(comp. (2.2)).

*Remark 3.* There are the fixed-point methods and other methods of the functional analysis used to study the nonlinear equations. What can be obtained by using these methods in the case of the nonlinear integral equations (1.3), (2.3)?

### 3. PROOF OF THE THEOREM 1

3.1. Let us remind that the following lemma holds true (see [6], (2.5); [7], (3.3)): for every integrable function (in the Lebesgue sense)  $f(x)$ ,  $x \in [\varphi_1(T), \varphi_1(T+U)]$  we have

$$(3.1) \quad \int_T^{T+U} f[\varphi_1(t)] \tilde{Z}^2(t) dt = \int_{\varphi_1(T)}^{\varphi_1(T+U)} f(x) dx, \quad U \in \left( 0, \frac{T}{\ln T} \right]$$

where

$$t - \varphi_1(t) \sim (1 - c)\pi(t),$$

$c$  is the Euler's constant and  $\pi(t)$  is the prime-counting function. In the case

$\mathring{T} = \varphi_1^{-1}(T)$ ,  $\widehat{T+U} = \varphi_1^{-1}(T+U)$  we obtain from (2.1)

$$(3.2) \quad \int_{\varphi_1^{-1}(T)}^{\varphi_1^{-1}(T+U)} f[\varphi_1(t)] \tilde{Z}^2(t) dt = \int_T^{T+U} f(x) dx.$$

3.2. By the simple formula

$$\int_0^a J_1(x) dx = 1 - J_0(a),$$

known from the theory of the Bessel's functions, we obtain

$$(3.3) \quad \int_{t_\nu}^{t_{\nu+1}} J_1(x) dx = J_0(t_\nu) - J_0(t_{\nu+1}).$$

Hence, from (3.3) by (3.2) the formula

$$(3.4) \quad \int_{\varphi_1^{-1}(t_\nu)}^{\varphi_1^{-1}(t_{\nu+1})} J_1[\varphi_1(t)] \tilde{Z}^2(t) dt = J_0(t_\nu) - J_0(t_{\nu+1})$$

is obtained.

3.3. It is also well-know that

$$(3.5) \quad J_\nu(x) = \sqrt{\frac{2}{\pi x}} \cos \left( x - \nu \frac{\pi}{2} - \frac{\pi}{4} \right) + \mathcal{O} \left( \frac{1}{x^{3/2}} \right), \quad x \rightarrow \infty$$

(the asymptotic formula for  $J_\nu(x)$ ). Since, by the Titchmarsh' formula (1.2)

$$(3.6) \quad \frac{1}{\sqrt{t_{\nu+1}}} = \frac{1}{\sqrt{t_\nu}} + \mathcal{O} \left( \frac{1}{t_\nu^{3/2} \ln t_\nu} \right),$$

it follows (see (3.5), (3.6)) that

$$(3.7) \quad \begin{aligned} J_0(t_\nu) - J_0(t_{\nu+1}) &= \\ &= \sqrt{\frac{2}{\pi t_\nu}} \left\{ \cos\left(t_\nu - \frac{\pi}{4}\right) - \cos\left(t_{\nu+1} - \frac{\pi}{4}\right) \right\} + \mathcal{O}\left(\frac{1}{t_\nu^{3/2}}\right). \end{aligned}$$

Next, (see (1.2))

$$(3.8) \quad \begin{aligned} \cos\left(t_\nu - \frac{\pi}{4}\right) - \cos\left(t_{\nu+1} - \frac{\pi}{4}\right) &= 2 \sin \frac{t_{\nu+1} - t_\nu}{2} \sin\left(\frac{t_{\nu+1} + t_\nu}{2} - \frac{\pi}{4}\right) = \\ &= 2 \sin \frac{t_{\nu+1} - t_\nu}{2} \sin\left(\frac{t_{\nu+1} - t_\nu}{2} + t_\nu - \frac{\pi}{4}\right) = \\ &= 2 \sin^2 \frac{t_{\nu+1} - t_\nu}{2} \cos\left(t_\nu - \frac{\pi}{4}\right) + \sin(t_{\nu+1} - t_\nu) \sin\left(t_\nu - \frac{\pi}{4}\right) = \\ &= \frac{2\pi}{\ln t_\nu} \sin\left(t_\nu - \frac{\pi}{4}\right) + \mathcal{O}\left(\frac{1}{\ln^2 t_\nu}\right). \end{aligned}$$

Hence, from (3.4) by (3.7), (3.8) the asymptotic formula

$$(3.9) \quad \begin{aligned} \int_{\varphi_1^{-1}(t_\nu)}^{\varphi_1^{-1}(t_{\nu+1})} J_1[\varphi_1(t)] \tilde{Z}^2(t) dt &= \\ &= \frac{2\sqrt{2\pi}}{\sqrt{t_\nu} \ln t_\nu} \sin\left(t_\nu - \frac{\pi}{4}\right) + \mathcal{O}\left(\frac{1}{\sqrt{t_\nu} \ln^2 t_\nu}\right) \end{aligned}$$

follows if the second condition in (1.4) is fulfilled. Then from (3.9) by the mean-value theorem (see (1.1), (1.4) and [19], (3.3)) we obtain

$$\begin{aligned} \int_{\varphi_1^{-1}(t_\nu)}^{\varphi_1^{-1}(t_{\nu+1})} J_1[\varphi_1(t)] \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 dt &= \\ &= \frac{2\sqrt{2\pi}}{\sqrt{t_\nu}} \sin\left(t_\nu - \frac{\pi}{4}\right) + \mathcal{O}\left(\frac{\ln \ln t_\nu}{\sqrt{t_\nu} \ln t_\nu}\right), \end{aligned}$$

i.e. the formula (1.5) holds true.

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